Linear Algebra: An Introduction

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Motivating Requirements of Linear Algebra

A Consistency check for Linear Algebra

1. Let a, b be two vectors. (Just like you learn in physics, for now).
2. $\langle a, b \rangle = a \cdot b = |a||b| \cos \theta$, where θ is the angle between the vectors a and b.

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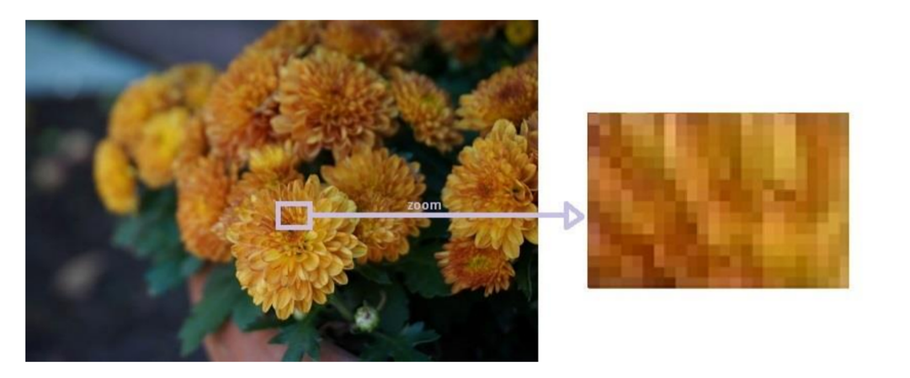
5. So, we have;

$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots\right) = \frac{(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)}{\sqrt{a_1^2 + \cdots + a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}}$$

An Application of Linear Algebra



Reference: Digital Image Matrix Operations by Williams Orenda

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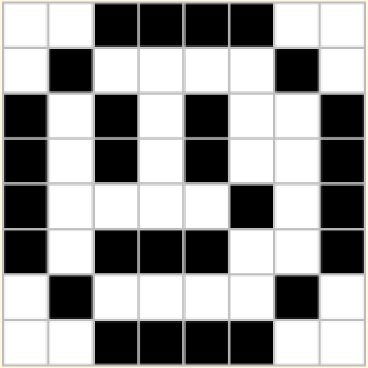

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

An Application of Linear Algebra

1. A Grayscale image of 512×480 pixels is represented by a matrix of order 512×480.
2. A color image of 512×480 pixels is represented by three matrices of order 512×480. We call these matrices as channels. There are Red, Green, Blue channels.
3. Now, we do like put the third matrix at first position, and the first matrix at third position. Hence, the high values of red matrix, now denotes the high value of blue matrix.

An Application of Linear Algebra

Figure 4: red-blue swap

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Building Blocks of Linear Algebra

Five W's about Numbers

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2. Where does number comes from?
3. Why are numbers required at all?
4. How numbers solve these problems?
5. When we should think about numbers?

Numbers vs Tuple of Numbers

Field $\mathcal{F}$

1. Two operations $+, \cdot$.
2. $a + b = b + a$ and $ab = ba$.
3. $a(b + c) = ab + ac$.
4. $a + (b + c) = (a + b) + c$.
5. $\exists 0$ s.t. $a + 0 = a$.
6. $\forall a \exists b$ s.t. $a + b = 0$.
7. $\exists 1$ s.t. $a \cdot 1 = a$.
8. $\forall a \neq 0 \exists b'$ s.t. $a \cdot b' = 1$.
9. $0 \neq 1$.

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Vector Space $\mathcal{V}$ over field $\mathcal{F}$

1. Two operations $\oplus, \odot$.
2. $u \oplus v = v \oplus u$.
3. $u \oplus (v \oplus w) = (u \oplus v) \oplus w$.
4. $\exists 0_{\mathcal{V}}$, s.t. $v \oplus 0_{\mathcal{V}} = v$.
5. $\forall v \exists u$ s.t. $v \oplus u = 0_{\mathcal{V}}$.
6. $\forall \alpha \in \mathcal{F}$,
   $\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)$.
7. $\forall \alpha, \beta \in \mathcal{F}$,
   $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$.
8. $(\alpha\beta) \odot v = \alpha \odot (\beta \odot v)$.
9. $1_{\mathcal{F}} \odot v = v$.

Spanning set in 2D Vector Space

1. There is a special car, which goes only up-down.

$$\downarrow\uparrow$$

Can you reach anywhere on the 2d plane with it?

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3. The car goes up-down, left-right, and also cornerwise.

   $$\downarrow\uparrow\rightleftarrows\nearrow\swarrow\ \nwarrow\searrow$$

   Can you reach anywhere on the 2d plane with it?

Spanning Set

Definition

A set of vectors $\{v_1, v_2, \ldots v_k\}$ is said to span the whole vector space $\mathcal{V}$, if for any vector $v \in \mathcal{V}$, there exists some scalers $\alpha_1, \alpha_2, \ldots \alpha_k \in \mathcal{F}$, such that,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

This is called a linear combination.
In the previous example, the first set of vectors was not spanning, but the second and third was.

Linear Independence

Consider a bird returning to its home. It visualizes the world as a 3d space.[1] For now, let x is the axis straightways, y is the axis sideways, and z is the axis from top of the sky to bottom of the ground.

1. Suppose the bird knows how to fly straight and sideways. Can it fly downwards or upwards?

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2. Suppose the bird knows how to fly straight and sideways. Can it fly parallel $x = y$ plane?

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Linearly Independent Set

Definition

A set of vectors $\{v_1, v_2, \dots v_k\}$ is said to be linearly independent, if for any v_i, cannot be written as a linear combination of the other vectors in the set, i.e. one cannot find some scalers $\alpha_1, \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_k \in \mathcal{F}$ such that,

$$v_i = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_k v_k$$

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$$v_i = \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_{i+1} v_{i+1} + \cdots + \alpha_k v_k$$

OR A set of vectors $\{v_1, v_2, \dots v_k\}$ is said to be linearly independent if there does not exist some scalers $\alpha_1, \alpha_2, \dots \alpha_k \in \mathcal{F}$, such that,

$$0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

where atleast one $\alpha_i \neq 0$.

Two Basic Results

Result (Extension of Spanning Set)

If S is a spanning set, then any superset of S is also a spanning set.

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If S is a spanning set, then any superset of S is also a spanning set.

Result (Deduction of Independent Set)

If S is a linearly independent set, then any subset of S is also a linearly independent set.

Basis of a Vector Space

Definition

A basis of $\mathcal{V}$ is a spanning set for $\mathcal{V}$ which is also linearly independent.

Exercise

1. *Show that, a basis of $\mathcal{V}$ is the smallest possible spanning set of $\mathcal{V}$.*
2. *Show that, a basis is the largest possible linearly independent set.*

The dimension of vector space

Theorem

If B_1 and B_2 are two bases of a vector space $\mathcal{V}$, then $|B_1| = |B_2|$, i.e. their sizes are same.

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To prove this, let, $B_1 = \{v_1, v_2, \ldots v_m\}$ and $B_2 = \{u_1, u_2, \ldots u_n\}$. However, We need another lemma.

Lemma (Replacement Theorem)

There exists v_i such that, $B_1^{(1)} = \{u_1, v_1, v_2, \ldots v_{i-1}, v_{i+1}, \ldots v_m\}$, which is obtained by including u_1 and removing v_i, is also a basis.

Proof of Basis Theorem

Proof.

Assume $m \neq n$, we shall use proof by contradiction. Without loss of generality, $m < n$.
Consider, replacing u_1 to get, $B_1^{(1)}$, which is a basis. Next, replace u_2 to $B_1^{(1)}$ to get the new basis, $B_1^{(2)}$.
Continue replacing all elements from B_2, until $B_1^{(k)}$ is completely filled with u_i's only, (i.e. all v_i's are removed).
But, this final basis is maximal linear independent set, but is a strict subset of B_2. B_2 cannot be basis. □

Example 1

Vector space $\mathbb{R}^n$, which is the space of n-dimensional vector.

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix} \right\}$$

is the Euclidean basis.
To express an element $x \in \mathbb{R}^n$, we use;

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}$$

Identify x as the tuple of the coefficients $(x_1, x_2, \dots x_n)$.

Example 2

Vector space $\mathbb{R}^2$, which is the space of 2 dimensional vector with real elements.

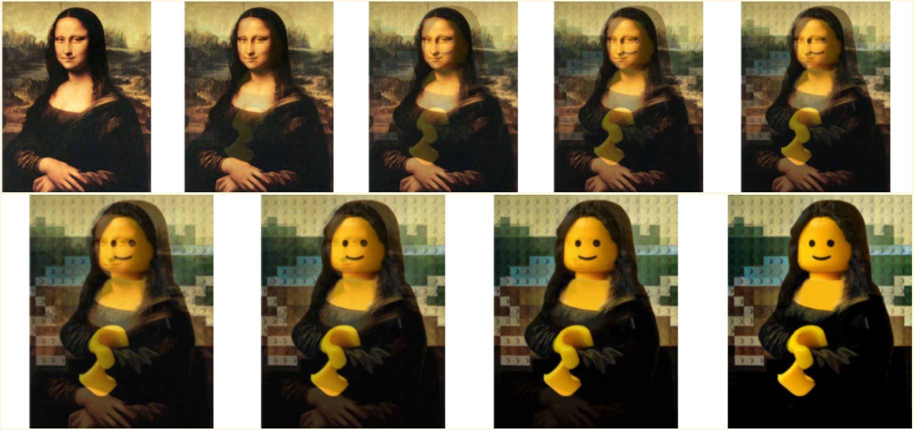
$$B = \left\{ \left(1, 1\right), \left(1, 2\right) \right\}$$

is a basis.
To express the usual vector $x = (4, 5)$, with respect to this basis,

$$x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

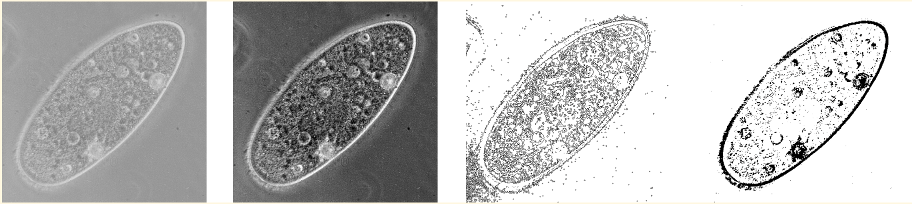
So identify x as $(3, 1)$, the tuple of coefficients.

Application of taking Linear Combinations



Matrix as a Linear Transformation

Application of identifying matrix as function



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1. $\mathcal{U}$ and $\mathcal{V}$ are both vector spaces over same field $\mathcal{F}$.
2. $f(\alpha v_1 \oplus \beta v_2) = \alpha f(v_1) \oplus \beta f(v_2)$, for any $\alpha, \beta \in \mathcal{F}$ and $v_1, v_2 \in \mathcal{U}$.

Examples of Linear Transformation

Example

Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be a function such that,

$$f(v) = \begin{cases} (1,0) & \text{if there are odd number of positive entries in v} \\ (0,1) & \text{otherwise} \end{cases}$$

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Example

Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be a function such that,

$$f(v) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where x_1 is sum of odd position elements, x_2 is sum of even position elements. Is it a linear transformation?

What's so good about it?

Let, $f : \mathcal{U} \to \mathcal{V}$ be a linear transformation. Let, $A = \{u_1, u_2, \dots u_m\}$ be the basis of $\mathcal{U}$.
Then,

$$u \in \mathcal{U} \implies u = \sum_i \alpha_i u_i$$

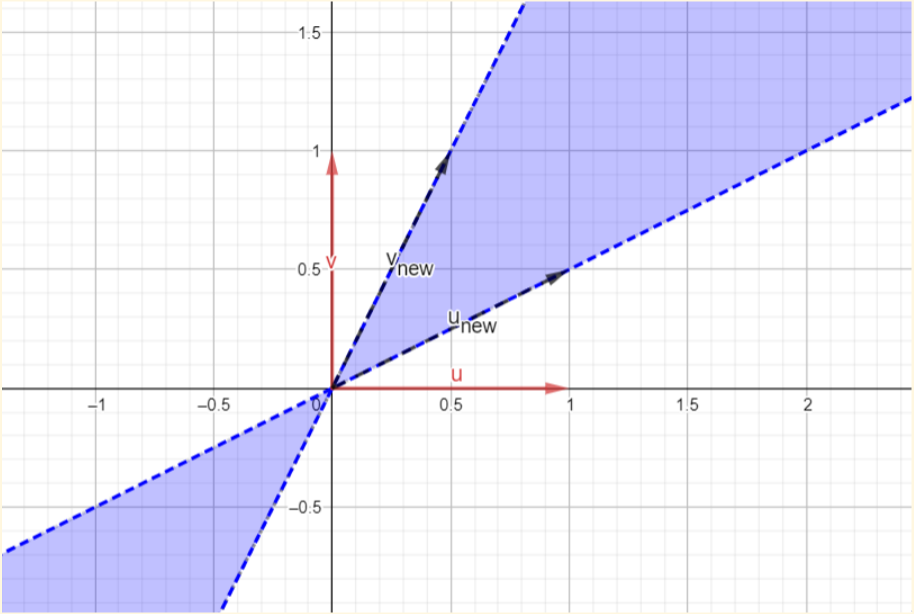
So,

$$f(u) = f\left(\sum_i \alpha_i u_i\right) = \sum_i \alpha_i f(u_i)$$

Result

To specify a linear transformation, it is just enough to know its action on the basis.

Example



Linear Transformation to Matrix

Now, let $B = \{v_1, v_2, \ldots v_n\}$ be a basis of vector space $\mathcal{V}$. Then, there are scalars, r_{ij}, $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$ such that,

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$$
\begin{aligned}
f(u_1) &= r_{11} v_1 + r_{12} v_2 + \cdots + r_{1n} v_n \\
f(u_2) &= r_{21} v_1 + r_{22} v_2 + \cdots + r_{2n} v_n \\
\dots\dots \\
f(u_m) &= r_{m1} v_1 + r_{m2} v_2 + \cdots + r_{mn} v_n
\end{aligned}
$$

We can now collect all these numbers r_{ij} together, so that we get an array of scalars, with m rows and n columns. This is **MATRIX** over the scalar field $\mathcal{F}$, corresponding to the transformation $f : U \to V$ with respect to the basis A and B.

Generalizing Matrix Vector Product

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5. So, we have the following transformation;

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_m \end{pmatrix} \to \begin{pmatrix} \sum_i r_{i1}\alpha_i \\ \sum_i r_{i2}\alpha_i \\ \dots \\ \sum_i r_{in}\alpha_i \end{pmatrix} = \begin{bmatrix} r_{11} & r_{21} & \dots & r_{m1} \\ r_{12} & r_{22} & \dots & r_{m2} \\ \vdots & \ddots & \vdots & \vdots \\ r_{1n} & r_{2n} & \dots & r_{mn} \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_m \end{pmatrix}$$

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6. $f(u_{m\times 1})_{n\times 1} = (R_f)_{n\times m} u_{m\times 1}$. Note the transpose of matrix.

Example 1

Example

$f : \mathbb{R}^4 \to \mathbb{R}^2$ such that,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_3 \\ x_2 + x_4 \end{pmatrix}$$

is given by the matrix;

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$$

Example 2

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2$ such that,

$$f(z) = ze^{i\theta}$$

where $z \in \mathbb{C}$. The transformation matrix works as follows;

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

This type of rotational transformation is denoted by a matrix of special property, these are called **Orthogonal** matrices.

Some more results

Theorem

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Exercise

1. *How does the transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = x$ looks like as matrix? Is it identity matrix?*
2. *What is the matrix corresponding to the transformation fg, i.e. $(fg)(u) = f(u)g(u)$.*
3. *What is the matrix corresponding to f/g? Is it AB^{-1}? or $B^{-1}A$?*
4. *What is the linear transformation corresponding to the matrix A^T? What is corresponding to $A \otimes B$, where $\otimes$ is the elementwise product.*

Kernel, Null Space and Range

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Definition

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Exercise

1. *If $f(x_0) = y_0$, then all solutions to the equation $f(x) = y_0$ is the set $x_0 + Ker(f)$.*
2. *Show that, $Ker(f)$ is a vector space.*

An Interesting Example

Let there are n cities, $C_1, C_2, \ldots, C_n$. Between the cities, we have one-way roads. Consider a $n \times n$ matrix R such that,

$$R_{ij} = \begin{cases} 1 & \text{if there is road from city i to city j} \\ 0 & \text{otherwise} \end{cases}$$

Then, consider square of this matrix.

$$(R^2)_{ij} = \sum_k R_{ik} R_{kj}$$

A term in the sum is 1 iff there is a road from city i to city j, through city k. So, $(R^2)_{ij}$ is the number of ways to reach city j from city i visting a city in between.
In other words, $(R + R^2 + \cdots + R^s)_{ij}$ is the number of ways to reach city i from city j in atmost s steps.

Building Row Rank and Column Rank

Consider a matrix A of order $m \times n$.

Definition

Let $\mathcal{R}(A)$ denote the span of the row vectors in A. Then, row rank of a matrix is the dimension of that spanning vector space.

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Consider a matrix A of order $m \times n$.

Definition

Let $\mathcal{R}(A)$ denote the span of the row vectors in A. Then, row rank of a matrix is the dimension of that spanning vector space.

Definition

Let $\mathcal{C}(A)$ denote the span of the column vectors in A. Then, column rank of a matrix is the dimension of that spanning vector space.

Some Results

Theorem

$$\textit{Row rank} \leq \min\{m, n\}$$

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Exercise

Show that, column space of a matrix $A = \mathcal{C}(A)$ *is same as the range space* $R(f)$.

Rank Theorems

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This common value is called Rank of a matrix.

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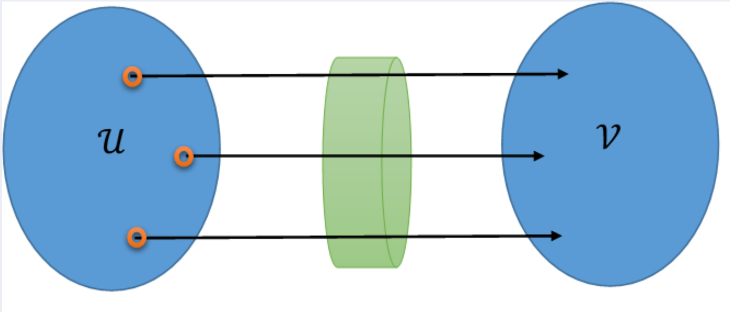
Proof.



□

Rank Nullity Theorem

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If $f : \mathcal{U} \to \mathcal{V}$ is a linear transformation with corresponding matrix A, then;

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Proof.

Let, $B = \{v_1, v_2, \dots v_n\}$ be a basis of $\mathcal{V}$. Some of these form the basis of $R(f)$. The rests are like useless fellows.
Enough to show, the number of such useless fellows is same as the dimension of $d(Ker(f))$ □

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There must be a vector $v \in \mathcal{F}^n$ such that, $B \cup \{v\}$ is linearly independent, hence is basis of $\mathcal{F}^n$.

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There must be a vector $v \in \mathcal{F}^n$ such that, $B \cup \{v\}$ is linearly independent, hence is basis of $\mathcal{F}^n$.
But, v does not have a pre-image, as $v \notin \mathcal{R}(f)$. □

Determinant of a Matrix

Working on the Determinant

Consider the matrix,

$$A = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$$

That means, it is a transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ such that,
$(1,0) \to (x_1, x_2)$ and $(0,1) \to (x_3, x_4)$.
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So the unit square goes to the parallelogram given by the side vectors (x_1, x_2) and (x_3, x_4). **Question:** What is the area of the parallelogram?

Determining formula for Determinant

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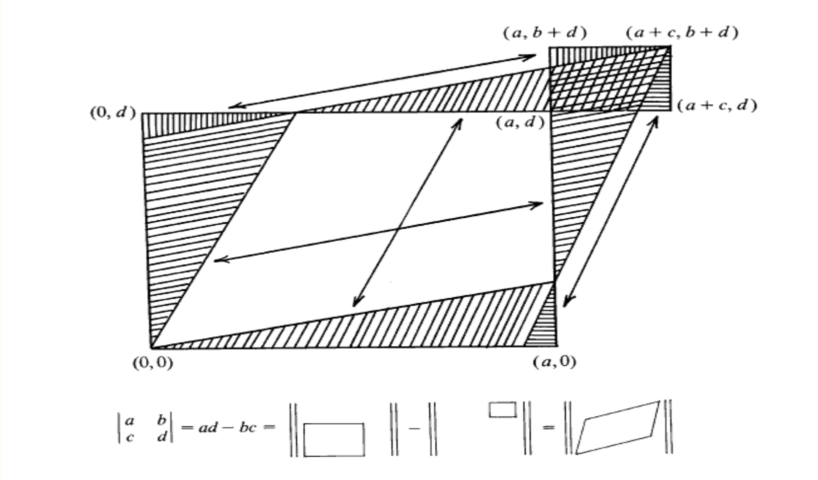

Determinant as Hypervolume

The answer to previous question is $\det(A) = x_1x_4 - x_2x_3$.
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1. Hence, you cannot think of determinant of non square matrix, since specifying the Hyperparallelopiped uniquely, you require atleast all n columns for $\mathbb{R}^n$.
2. If $\det(A) = 0$, then the linear transformation is not invertible. Think that the linear transformation is squeezing many vectors between two of its Hyperspace, at a single Hyperplane, which makes it impossible to retrace back exactly from where these vectors come from.

Properties of Determinant

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4. $\det(A^{-1}) = \dfrac{1}{\det(A)}$. Since, if unit cube maps to something of volume $\det(A)$ in range space, an unit cube (or volume) in range space maps back to something of volume $1/\det(A)$, simple unitary method.

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So, an unit cube in $\mathcal{U}$ becomes of volume $\det(A)\det(B)$ in $\mathcal{W}$, when we apply $f \circ g$, whose corresponding matrix is AB. □

Inner Product Space

Inner Product

1. $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathcal{F}$.
2. $\langle x, y \rangle = \langle y, x \rangle$.
3. $\langle ax, y \rangle = a\langle x, y \rangle$.
4. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$.
5. $\langle x, x \rangle > 0 \quad \forall x \neq 0_{\mathcal{V}}$.

We call $\sqrt{\langle x, x \rangle} = \|x\|$, the norm of x.

Inner Product vs Metric

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Distance function or Metric

1. $d(\cdot, \cdot) : S \times S \to \mathbb{R}$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \geq d(x, z) + d(z, y)$.
4. $d(x, x) > 0 \quad \forall x \in S - \{0\}$

Matrix of Inner Product

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3. Let, $B = \{v_1, v_2, \dots v_n\}$ be a basis of $\mathcal{V}$.
4. Let, A be a matrix such that,

$$(A)_{ij} = \langle v_i, v_j \rangle$$

Note that, A is symmetric matrix, $A^\top = A$.

Building Inner Product Formula

1. We have, $\mathrm{x} = (x_1, x_2, \ldots x_n)^\top$ and $\mathrm{y} = (y_1, y_2, \ldots y_n)^\top$.
2. Therefore,

$$\begin{aligned}
\langle \mathrm{x}, \mathrm{y} \rangle = \langle \sum_i x_i v_i, \sum_j y_j v_j \rangle &= \sum_i \langle x_i v_i, \sum_j y_j v_j \rangle \\
= \sum_i x_i \langle v_i, \sum_j y_j v_j \rangle &= \sum_i x_i \langle \sum_j y_j v_j, v_i \rangle \\
= \sum_i x_i \sum_j y_j \langle v_j, v_i \rangle &= \sum_i \sum_j x_i y_j \langle v_i, v_j \rangle \\
&= \sum_i \sum_j x_i A_{ij} y_j \\
&= \mathrm{x}^\top A \mathrm{y}
\end{aligned}$$

This A is called the inner product basis matrix. The forms $x^\top A y$ are called **Bilinear** forms.

Formula Inner Product

1. Let, B be the usual n dimensional Euclidean basis. Let, e_i be the vector with all zeros except a one at i-th position.
2. Define, inner product basis matrix to be identity matrix, as usual.
3. Then,

$$\langle x, y \rangle = \sum_i x_i \delta_{ij} y_j$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Hence, $\langle x, y \rangle = \sum_i x_i y_i$.

Inner Product to Projection

1. Let's say, you want to find the projection (or the component) of x that is aligned with e_1, i.e. what should be the projection of $(x_1, x_2, \dots x_n)$ onto $(1, 0, 0, \dots 0)$?
2. What should be the projection of $(x_1, x_2, \dots x_n)$ onto $(\alpha, 0, 0, \dots 0)$?
3. What should be the projection of $(x_1, x_2, \dots x_n)$ onto $(y_1, y_2, \dots y_n)$?

Inner Product to Projection (Contd.)

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4. So, you consider;

$$\begin{aligned} x &= \alpha y + \alpha_1 y_1 + \alpha_2 y_2 + \ldots \alpha_{n-1} y_{n-1} \\ \Rightarrow \langle x, y\rangle &= \alpha \langle y, y\rangle + \alpha_1 \langle y_1, y\rangle + \cdots + \alpha_{n-1} \langle y_{n-1}, y\rangle \\ \Rightarrow \langle x, y\rangle &= \alpha \langle y, y\rangle \\ \Rightarrow \frac{\langle x, y\rangle}{\langle y, y\rangle} &= \alpha \end{aligned}$$

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- Note that, geometrically, it is easy to see, the projection of x onto y is $x \cos\theta$.
- From the discussion before, the projection was αy, and hence, $x \cos\theta = \alpha y$.
- Therefore, taking norm and equating,

$$\|x \cos\theta\| = \|x\| \cos\theta = \frac{\langle x, y\rangle}{\langle y, y\rangle} \|y\|$$

and we end up,

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\| \|y\|}$$

Eigen Values

Eigen Value as Projection

- In a projection of x onto y, we try to find the best part of x which can be made parallel to y.
- Eigenvalues are kind of projection of a linear transformation. You kind of retain best parts of a linear transformation, which explains them most.

Definition

A scalar $\lambda \in \mathcal{F}$ is said to be an eigenvalue with corresponding eigenvector $v \in \mathcal{V}$ of the matrix A if;

$$Av = \lambda v$$

with $v \neq 0$.

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Exercise

Find eigenvalues of a diagonal matrix with entries $a_1, a_2, \ldots a_n$.

Find eigenvalues of the matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Decomposition with Eigen values

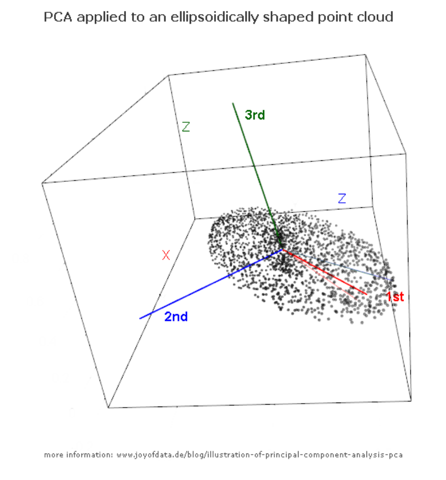
Let, B be a basis with $v_1, v_2, \dots v_n$, each of which is an eigenvector, with corresponding eigenvalue $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.
Then,

$$\begin{aligned} Ax &= A(\sum_i \alpha_i v_i) \\ &= \sum_i \alpha_i A v_i \\ &= \sum_i \alpha_i \lambda_i v_i \\ &\approx \alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \cdots + \alpha_k\lambda_k v_k \end{aligned}$$

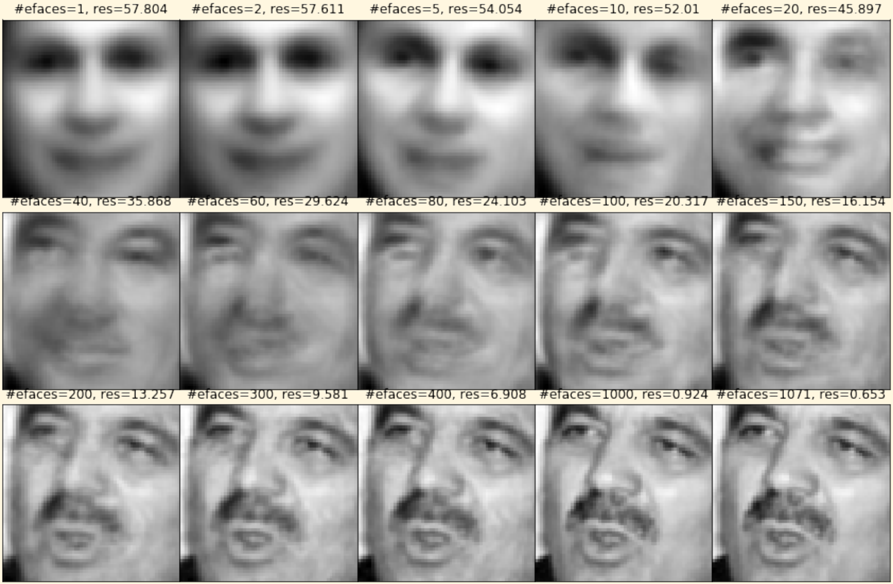
where $k << n$.
Instead of remembering n^2 numbers to specify A, we can simply remember $k(n+1)$, numbers (kn for the eigenvectors and k many for eigenvalues).

Example: Principal Component Analysis

Reference: Math stack exchange.



Example: Eigen Face (from Sandipanweb Wordpress)



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